Differential Geometry and its Applications 14 (2001) 209–216
North-Holland
www.elsevier.com/locate/difgeo

Lower bounds for contraction constants of non-zero degree mappings onto the sphere[1]

Eleonora Ciriza
Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", 00133 Roma, Italia

Marcelo Llarull[2]
Department of Mathematics, William Paterson University, Wayne, New Jersey 07470, USA

Communicated by J.P. Bourguignon
Received 28 February 2000

Abstract: This paper studies contraction constants of non-zero degree mappings from compact spin Riemannian manifolds onto the standard Riemannian sphere. Assuming uniform lower bound for the scalar curvature, we find a sharp lower bound for the dilation constants in terms of the dimension of the sphere. In the best case, we prove rigidity.

Keywords: Lower bounds, scalar curvature, spin manifold.

MS classification: 53C20, 53C21.

1. Introduction

Let M and N be compact Riemannian manifolds of dimension n with metrics g and $\tilde{g}$, respectively. A map $f : M \to N$ is ϵ*-contracting* ($\epsilon > 0$) if

$$\| f_* v \|_{\tilde{g}} \leqslant \epsilon \| v \|_g \tag{1}$$

for all v tangent vectors to M. The constant ϵ is called a *contraction constant*.

Notice that if f is ϵ-contracting, then it is c-contracting for all $c \geqslant \epsilon$ and that contracting mappings could contract or stretch lengths depending on whether the contraction constant is <1 or >1, respectively.

The *degree* of $f : M \to N$ between compact Riemannian manifolds is (see [3])

$$\deg(f) = \frac{\displaystyle\int_M f^* \omega}{\displaystyle\int_N \omega} \tag{2}$$

where ω is an n-form on N with non-zero integral.

We are interested in studying non-zero degree contracting maps into spheres, i.e., when $N = S^n$.

[1] Work partially supported by the CFR of William Paterson University and CNPq, Brasil.
[2] *E-mail*: llarullm@wpunj.edu

PII S0926-2245(01)00036-5

Let $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{k=1}^{k=n+1} x_k^2 = 1\}$ be the standard n sphere with the usual Riemannian metric g_0.

Given a map $f : M \to S^n$, we define,

$$\delta_f = \sup \frac{\| f_* v\|}{\|v\|} \tag{3}$$

where v runs through all non-zero tangent vectors to M. Then δ defines a map into $\mathbb{R}$, i.e.,

$$\delta : [M^n, S^n] \longrightarrow \mathbb{R}, \qquad f \longmapsto \delta_f. \tag{4}$$

By definition, any f is δ_f-contracting.

For a given Riemannian manifold M, is there a non-zero degree ϵ-contracting map $f : M \to S^n$ for any $\epsilon > 0$? If not, is there a lower bound for the contraction constants? How sharp is this bound?

2. Previous lower bounds

There are some results for the case when M and N are spheres of not necessarily the same dimensions. Let $f : S^m \to S^n$. R. Olivier [6] proved the following.

Theorem 1 (Olivier). *Let $f : S^m \to S^n$ be a differentiable mapping. If $m = n$ and f has even nonzero degree, or if $m > n = 2$ and f is not homotopic to zero, then $\delta_f > 2$.*

Using algebraic topology techniques, in [2], H.B. Lawson, jr. extended this result into the following.

Let $\{f\}$ denote the homotopy class of the function f in $\pi_m(S^n)$ and let $\Sigma : \pi_{m-1}(S^{n-1}) \to \pi_m(S^n)$ be the suspension homomorphism. Then,

Theorem 2 (Lawson). *Let $f : S^m \longrightarrow S^n$.*

(1) *If $m > n$ for any $n > 0$ and if f is not homotopic to zero, then $\delta_f \geqslant 2$.*

(2) *If $m = 2k > 0$, $2\{f\} \neq 0$ and $\{f\} \notin \Sigma(\pi_{2k-1}(S^{n-1}))$, then $\delta_f \geqslant 3$.*

In [1], introducing the scalar curvature, M. Gromov and H.B. Lawson, jr. established a lower bound for the contraction constant for non-zero degree functions $f : M \to S^n$, where M is a compact oriented Riemannian n-manifold, with $n \leqslant 7$.

Theorem 3 (Gromov–Lawson). *Let M be a compact oriented Riemannian n-manifold, $n \leqslant 7$, with scalar curvature $\geqslant 1$. Then, if $f : (M, \partial M) \to (S^n, *)$ has non-zero degree, then $\delta_f > \sqrt{n/(n-1)}(1/2^n\pi)$.*

R. Schoen and S.T. Yau have established similar results in general dimensions.

Even when in higher dimensions, the scalar curvature is such a weak measure of the geometry of a manifold, it imposes some restriction on the contraction constants. When the scalar curvature of the manifold M has a strictly positive lower bound, then M cannot admit arbitrarily contracting non-zero degree mappings onto the sphere.

This fact was used in [1] to show that *enlargeable* spin manifolds cannot carry a metric with positive scalar curvature.

In the next section we extend Theorem 3 to the case of a compact spin manifold of arbitrary dimension n.

3. Lower bounds for compact spin manifolds

A *spin manifold* is an oriented manifold that carries a spin structure on its tangent bundle, see [3].

Recall that a manifold M is spin if and only if the first and second Whitney classes of M are both zero. Thus, spheres are spin manifolds and the spin assumption is granted automatically.

For the rest of this paper M will be assumed to be a compact spin manifold. Suppose that $n = 2m$. Consider the following spin structure on (M, g),

$$S = P_{\mathrm{Spin}_n}(M) \times_\lambda \mathbb{C}\mathrm{l}_n$$

with the induced connection, where λ is the representation by left multiplication and $\mathbb{C}\mathrm{l}_n$ denotes the complexification of Cl_n.

S has a $\mathbb{Z}_2$-grading given by the oriented "volume element" ω. Fix $p \in M$ and choose a local pointwise orthonormal tangent vector fields around p, $\{e_1, e_2, \ldots, e_n\}$ such that $(\nabla e_k)_p = 0$. Then, $\omega = i^{n/2} e_1 \cdot e_2 \cdots e_n$, where $\cdot$ denotes Clifford multiplication.

Then S has the decomposition

$$S = S^+ \oplus S^-$$

into the $+1$ and -1 eigenvalues of Clifford multiplication by ω.

Since S^n is spin, when n is even, over (S^n, g_0) we can carry out the same construction to get the bundle

$$E_0 = P_{\mathrm{Spin}_n}(S^n) \times_\lambda \mathbb{C}\mathrm{l}_n$$

with the induced metric and connection from (S^n, g_0).

Fix $x \in S^n$ and choose a local pointwise orthonormal tangent vector fields around x, $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ such that $(\nabla \epsilon_k)_x = 0$. Then the "volume element" $\omega_0 = i^{n/2} \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_n$ gives the splitting $E_0 = E_0^+ \oplus E_0^-$, into the $+1$ and -1 eigenspaces of ω_0 .

Using f, we pull-back the vector bundle E_0 to the vector bundle $E = f^* E_0$ over (M, g), which as a bundle over M has also the splitting $E = E^+ \oplus E^- = f^* E_0^+ \oplus f^* E_0^-$.

Consider now the tensor product bundle $S \otimes E$ over M with the tensor product metric and connection, then

$$S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E).$$

In terms of an orthonormal basis of tangent vectors at p, the *twisted Dirac operator* of $S \otimes E$, $D_E : \Gamma(S \otimes E) \longrightarrow \Gamma(S \otimes E)$, is given by

$$D_E = \sum_{k=1}^{n} e_k \cdot \nabla_{e_k}.$$

It is easy to see that $D_E(\Gamma(S\otimes E^+)) \subseteq \Gamma(S\otimes E^+)$. Let $D_{E^+} = D_E|_{S\otimes E^+}$ denote the restriction of D_E to $S\otimes E^+$.

The Dirac operator D_E satisfies the *Bochner–Lichnerowicz formula*, (see [3])

$$D_E^2 = \nabla^*\nabla + \tfrac{1}{4}\kappa + R^E, \tag{5}$$

where the operator $\nabla^*\nabla : \Gamma(S\otimes E) \longrightarrow \Gamma(S\otimes E)$ is defined in terms of a local basis of pointwise orthonormal tangent vector fields by

$$\nabla^*\nabla = -\sum_{k=1}^{n} \nabla_{e_k}\nabla_{e_k} + \nabla_{\nabla_{e_k}e_k}, \qquad \kappa = \sum_{i,j} g(\mathrm{R}_{e_ie_j}e_j, e_i)$$

is the scalar curvature of M, g is the Riemannian metric and R the curvature tensor of M.

Considering the inner product $\langle\cdot\,,\cdot\rangle$ on the space $\Gamma(S\otimes E)$ of cross-sections defined by $\langle\phi,\psi\rangle = \int_M g_x(\phi,\psi) \quad \forall\phi,\psi \in \Gamma(S\otimes E)$, we can write equation (5) as

$$\begin{aligned}\langle D_E^2\phi,\phi\rangle &= \langle\nabla^*\nabla\phi,\phi\rangle + \tfrac{1}{4}\kappa\langle\phi,\phi\rangle + \langle R^E\phi,\phi\rangle\\ &= \langle\nabla\phi,\nabla\phi\rangle + \tfrac{1}{4}\kappa \,\|\phi\|^2 + \langle R^E\phi,\phi\rangle.\end{aligned} \tag{6}$$

The curvature term is bounded below by, (see [5])

$$\langle R^E\phi,\phi\rangle \geqslant -\frac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}\,\|\phi\|^2 \tag{7}$$

where the scalars λ_i's satisfy pointwise $f_*e_i = (1/\lambda_i)\epsilon_i$, for $1 \leqslant i \leqslant n$.

Thus, equation (6) gives, therefore,

$$\langle D_E^2\phi,\phi\rangle \geqslant \frac{1}{4}\kappa\,\|\phi\|^2 - \frac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}\,\|\phi\|^2\,. \tag{8}$$

Remark 4. Since lower bounds of $\langle D_E^2\phi,\phi\rangle$ are given in terms of $\langle R^E\phi,\phi\rangle$, the key point is to twist the spin bundle with a convenient twisting factor to find a appropriate lower bound of $\langle R^E\phi,\phi\rangle$. In this case, the appropriate choice of the twisting bundle factor is E^+, which makes possible the lower estimate of $\langle R^E\phi,\phi\rangle$ given by equation (7) used to obtain the sharp results.

Now we are ready for the main result in this paper.

Theorem 5. *Let (M,g) be a compact spin manifold of dimension n with metric g and scalar curvature $\kappa_g \geqslant 1$. If $f : (M,g) \to (S^n, g_0)$ is a non-zero degree ϵ-contracting mapping, then $\epsilon \geqslant 1/\sqrt{n(n-1)}$; in particular, $\delta_f \geqslant 1/\sqrt{n(n-1)}$. Moreover, if $\epsilon = 1/\sqrt{n(n-1)}$, then f is a dilation and $M \equiv S^n_{\sqrt{n(n-1)}}$ (an n-sphere of radius $\sqrt{n(n-1)}$).*

Proof. The theorem will be proved by contradiction. Suppose that there exists a non-zero degree ϵ-contracting mapping $f : (M,g) \to (S^n, g_0)$ with $\epsilon < 1/\sqrt{n(n-1)}$. Thus, the scalars λ_i's in (7) satisfy

$$\left|\frac{1}{\lambda_i}\right| < \frac{1}{\sqrt{n(n-1)}} \tag{9}$$

and therefore,

$$\left|\frac{1}{\lambda_i\lambda_j}\right| < \frac{1}{n(n-1)}$$

combining with equation (7), we obtain

$$\langle R^E\phi,\phi\rangle \geqslant -\tfrac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}\ \|\phi\|^2 > -\tfrac{1}{4}\sum_{i\neq j}\frac{1}{n(n-1)}\ \|\phi\|^2 > -\tfrac{1}{4}\ \|\phi\|^2 \tag{10}$$

and equation (6) becomes

$$\langle D_E^2\phi,\phi\rangle > \tfrac{1}{4}(\kappa-1)\ \|\phi\|^2 \geqslant 0. \tag{11}$$

Therefore, if $\kappa \geqslant 1$, but not identically 1, then $0 = \ker(D_E^2) = \ker(D_E)$ and $\ker(D_{E^+}) = 0$ (see [1]). Since $\ker(D_{E^+}) = \ker(D_{E^+}^+) \oplus \ker(D_{E^+}^-)$, then $\ker(D_{E^+}^+) = 0 = \ker(D_{E^+}^-)$. The index of $D_{E^+}^+$ is given by

$$\mathrm{Index}(D_{E^+}^+) = \dim(\ker(D_{E^+}^+)) - \dim(\ker(D_{E^+}^-)) = 0. \tag{12}$$

However, this index is not zero. The Atiyah–Singer Index Theorem gives that

$$\mathrm{Index}(D_{E^+}^+) = \{\mathrm{ch}(E^+)\hat{A}(M)\}[M],$$

where $\mathrm{ch}(E^+)$ is the Chern character of E^+ and $\hat{A}$ is the total $\hat{A}$-class of M.

A compact spin manifold with $\kappa > 0$ has $\hat{A}(M) = 1$. In this case, the index is given by (see [3])

$$\mathrm{Index}(D_{E^+}^+) = \frac{1}{(\frac{1}{2}n-1)!}\ \deg(f)\int_{S^n} c_{n/2}(E_0^+) \neq 0 \tag{13}$$

since $\deg(f) \neq 0$ and the top Chern class $c_{n/2}(E_0^+) \neq 0$ on the even dimensional sphere because it is a non-zero multiple of the Euler number.

If $\kappa \equiv 1$, then re-writing $\kappa = \sum_{i\neq j}^n (1/n(n-1))$ and replacing it in equation (8), we obtain

$$\langle D_E^2\phi,\phi\rangle \geqslant \frac{1}{4}\sum_{i\neq j}^n\left[\frac{1}{n(n-1)} - \frac{1}{\lambda_i\lambda_j}\right]\|\phi\|^2 \geqslant 0. \tag{14}$$

Since $\mathrm{Index}(D_E) \neq 0$, then $\ker(D_E) \neq 0$ and there exists $0 \neq \phi \in \Gamma(S\otimes E)$ with $D_E(\phi) = 0$ and

$$0 \geqslant \frac{1}{4}\sum_{i\neq j}^n\left[\frac{1}{n(n-1)} - \frac{1}{\lambda_i\lambda_j}\right]\|\phi\|^2 \geqslant 0.$$

By (9), each term in the sum is non-negative, so $1/n(n-1) - 1/\lambda_i\lambda_j = 0$ and $\lambda_i\lambda_j = n(n-1)$ for $1 \leqslant i, j \leqslant n$. Thus, $\lambda_i \equiv \sqrt{n(n-1)}$ and f is a dilation with constant $1/\sqrt{n(n-1)}$ and $M \equiv S^n_{\sqrt{n(n-1)}}$.

If $\dim(M) = n = 2m-1$, odd dimensional, we consider the (even) $(n+1)$-dimensional spin manifold $M \times S_r^1$, with $r > 1$, and the function

$$M \times S_r^1 \xrightarrow{f\times(1/r)\mathrm{id}} S^n \times S^1 \xrightarrow{h} S^n \wedge S^1 \cong S^{n+1}$$

where S^1_r is the one dimensional sphere of radius r, $f \times (1/r)\,\mathrm{id}$ is defined as $(f \times (1/r)\,\mathrm{id})(p,t) = (f(p), t/r)\,\forall (p,t) \in M \times S^1$, and where h is a 1-contracting map into the smash product of non-zero degree. Therefore, $f \times (1/r)\,\mathrm{id}$ is $1/r$-contracting on the last factor and $1/\sqrt{n(n-1)}$-contracting on the first factor.

Now we construct the same spinor bundles as in the even dimensional case and find the appropriate lower bound for the curvature term in equation (8).

Without loss of generality, we can assume that the subindex $n+1$ corresponds to the S^1_r factor. Assuming that f is $1/\sqrt{n(n-1)}$-contracting, the corresponding equations to (9) are

$$\left|\frac{1}{\lambda_i}\right| \leqslant \frac{1}{\sqrt{n(n-1)}} \qquad \text{for} \quad 1 \leqslant i,j \leqslant n \quad \text{and} \quad \left|\frac{1}{\lambda_{n+1}}\right| \leqslant \frac{1}{r}.$$

Separating the terms arising from M and from S^1_r in equation (7), the curvature term can be bounded as follows:

$$\langle R^E\phi,\phi\rangle \geqslant -\frac{1}{4}\sum_{i\neq j}^{n}\frac{1}{\lambda_i\lambda_j}\,\|\phi\|^2 - 2\,\frac{1}{4}\sum_{i=1}^{n}\frac{1}{\lambda_{n+1}\lambda_i}\,\|\phi\|^2, \tag{15}$$

$$\langle R^E\phi,\phi\rangle \geqslant -\frac{1}{4}\sum_{i\neq j}^{n}\frac{1}{n(n-1)}\,\|\phi\|^2 - \frac{1}{2}\sum_{i=1}^{n}\frac{1}{r\sqrt{n(n-1)}}\,\|\phi\|^2,$$

$$\langle R^E\phi,\phi\rangle \geqslant -\frac{1}{4}\,\|\phi\|^2 - \frac{1}{2}\,\frac{n}{r\sqrt{n(n-1)}}\,\|\phi\|^2$$

which gives the following bound for equation (6)

$$\begin{aligned}\langle D_E^2\phi,\phi\rangle &\geqslant \frac{1}{4}\kappa\,\|\phi\|^2 - \frac{1}{4}\,\|\phi\|^2 - \frac{1}{2}\,\frac{n}{r\sqrt{n(n-1)}}\,\|\phi\|^2\\ &\geqslant \frac{1}{4}\left[\kappa - 1 - \frac{2n}{r\sqrt{n(n-1)}}\right]\|\phi\|^2 .\end{aligned}$$

If $\kappa \geqslant 1$, but not identically 1, since r is arbitrarily > 1, we conclude as before that $\ker(D_E) = 0$ and therefore, $\mathrm{Index}(D^+_{E^+}) = 0$ which is a contradiction since by the Atiyah Index Theorem $\mathrm{Index}(D^+_{E^+}) \neq 0$.

If $\kappa \equiv 1$, then rewriting $\kappa = \sum_{i\neq j}^{n}(1/n(n-1))$ and using (15), equation (8) gives

$$\langle D_E^2\phi,\phi\rangle \geqslant \frac{1}{4}\sum_{i\neq j}^{n}\left[\frac{1}{n(n-1)} - \frac{1}{\lambda_i\lambda_j} - \frac{2n}{r\sqrt{n(n-1)}}\right]\|\phi\|^2 \geqslant 0.$$

Since $\mathrm{Index}(D_{E^+}) \neq 0$, there is a non-zero harmonic spinor which gives

$$0 \geqslant \frac{1}{4}\sum_{i\neq j}^{n}\left[\frac{1}{n(n-1)} - \frac{1}{\lambda_i\lambda_j} - \frac{2n}{r\sqrt{n(n-1)}}\right]\|\phi\|^2 \geqslant 0.$$

Thus, as before, $\lambda_i \equiv \sqrt{n(n-1)}$ and f is a dilation and $M \equiv S^n_{\sqrt{n(n-1)}}$.

Therefore, $\epsilon \geqslant 1/\sqrt{n(n-1)}$ and by equation (3),

$$\delta_f \geqslant \frac{1}{\sqrt{n(n-1)}}.$$

Remark 6. The lower bound for δ_f given by Theorem 5 is sharp.

Consider the identity function $\mathrm{Id} : (S^n, g) \to (S^n, g_0)$, where the metric on the first copy of S^n is $g = n(n-1)\, g_0$. Then, the scalar curvature of (S^n, g) is $\kappa_g \equiv 1 \geqslant 1$ and the identity is $1/\sqrt{n(n-1)}$-contracting. In fact,

$$\|v\|_{g_0} = \frac{1}{\sqrt{n(n-1)}}\sqrt{n(n-1)}\,\|v\|_{g_0} = \frac{1}{\sqrt{n(n-1)}}\|v\|_g$$

and $\delta_{\mathrm{Id}} \equiv 1/\sqrt{n(n-1)}$.

Under the spin hypothesis, Theorem 5 improves the lower bounds and extends Theorem 3.

Introducing the $\hat{A}$-degree (see [3]), we can extend Theorem 3 to the case when $\dim(M) - \dim(S^n) = 4k$ for $k \geqslant 0$.

Let M and N be compact Riemannian manifolds such that $\dim(M) - \dim(N) = 4k \geqslant 0$. Let ω be the volume form on N with non-zero integral and let $\hat{A}_k(M)$ be a de Rham representative (a closed $4k$-form) of the kth component of the total $\hat{A}$-class of M. Then the *$\hat{A}$-degree of f* is

$$\hat{A}\text{-}\deg(f) = \frac{\displaystyle\int_M f^*\omega \wedge \hat{A}_k(M)}{\displaystyle\int_N \omega}. \tag{16}$$

Theorem 7. *Let (M^{n+4k}, g) be a compact spin manifold of dimension $n + 4k$ with metric g and scalar curvature $\kappa_g \geqslant 1$. If $f : (M^{n+4k}, g) \to (S^n, g_0)$ is a non-zero $\hat{A}$-degree ϵ-contracting mapping, then $\epsilon \geqslant 1/\sqrt{n(n-1)}$; in particular, $\delta_f \geqslant 1/\sqrt{n(n-1)}$. Moreover, if $\epsilon \equiv 1/\sqrt{n(n-1)}$, then f is a dilating submersion.*

Proof. The proof is identical to that of Theorem 5. We need to notice the following.

(a) The lower bound for the curvature term (7) still applies since R^E is the curvature of a pull-back bundle over (S^n, g_0).

(b) The scalars λ_i's are such that $f_* e_i = (1/\lambda_i)\epsilon_i$ for $1 \leqslant i \leqslant n$ and $f_* e_i = 0$ for $n + 1 \leqslant i \leqslant n + 4k$. The scalars λ_i's still satisfy (9).

(c) In equation (6), κ stands for the scalar curvature of M^{n+4k}.

(d) All the above leads to equation (12).

(e) $\mathrm{Index}(D^+_{E^+}) = (1/(n/2-1)!)\int_M f^*\omega_n \hat{A}[M] = (c/(n/2-1)!)\,\hat{A}\text{-}\deg(f) \neq 0$, where $c \neq 0$ and ω_n is a generator of $H^n(S^n, \mathbb{Z})$.

(f) If $\kappa \equiv 1$, then κ can still be written as $\sum_{i\neq j}^n (1/n(n-1))$ and equation (14) is the same.

Notice that in the proofs of Theorems 5 and 7, the lower bound for the curvature term is found using bounds for the products of pairs of the scalars λ_i's. Therefore, if the function f is contracting on "2-planes", the same conclusions are true.

A function $f : M \to N$ is (Λ^2, ϵ)-*contracting* if $\| f^*\alpha \| \leqslant \epsilon \|\alpha\|$ for all 2-forms $\alpha \in \Lambda^2(N)$ (see [1]).

Corollary 8. *Let (M^{n+4k}, g) be a compact spin manifold of dimension $n + 4k$ with metric g and scalar curvature $\kappa_g \geqslant 1$. Let $f : (M^{n+4k}, g) \to (S^n, g_0)$ be a (Λ^2, ϵ)-contracting mapping. If $k = 0$ and f has non-zero degree, then $\epsilon \geqslant 1/\sqrt{n(n-1)}$; in particular, $\delta_f \geqslant 1/\sqrt{n(n-1)}$. Moreover, if $\epsilon \equiv 1/\sqrt{n(n-1)}$, then f is a dilating submersion. If $k > 0$ and f has non-zero $\hat{A}$-degree, then $\epsilon \geqslant 1/\sqrt{n(n-1)}$; in particular, $\delta_f \geqslant 1/\sqrt{n(n-1)}$.*

These conclusions are no longer valid if we require that f be contracting on k-planes with $k \geqslant 3$. For an explicit example see [4].

4. Lower bounds for non-compact spin manifolds

If M is not necessarily compact, we can modify the hypothesis as follows. A function f is *constant at infinity* if it is constant outside a compact set. In this case, (2) and (16) still make sense and the two concepts of degree can be extended for the non-compact case provided that the function f is constant at infinity. Therefore,

Theorem 9. *Let (M^{n+4k}, g) be a non-compact spin manifold of dimension $n + 4k$ with metric g and scalar curvature $\kappa_g \geqslant 1$ and consider an ϵ-contracting mapping, $f : (M^{n+4k}, g) \to (S^n, g_0)$, which is constant at infinity. If $k = 0$ and f has non-zero degree, or, if $k > 0$ and f has non-zero $\hat{A}$-degree, then $\epsilon \geqslant 1/\sqrt{n(n-1)}$. In particular, $\delta_f \geqslant 1/\sqrt{n(n-1)}$.*

Acknowledgment

M. Llarull would like to thank everyone at the *Instituto de Matemática Pura e Aplicada* (*IMPA*) for their hospitality and support during his visit.

References

[1] M. Gromov and H.B. Lawson, jr., Positive scalar curvature and the Dirac operator on compact Riemannian manifolds, *Inst. Hautes Études Sci. Publ. Math.* **58** (1984) 83–196.
[2] H.B. Lawson, jr., Stretching phenomena in mappings of spheres, *Proc. Amer. Math. Soc.* **19** (1968) 433–435.
[3] H.B. Lawson, jr. and M-L. Michelsohn, *Spin Geometry*, Princeton Math. Series 38 (Princeton University Press, Princeton, 1989).
[4] M. Llarull, Scalar curvature and volume of a Riemannian manifold, *Geom. Dedicata* **56** (1995) 1–4.
[5] M. Llarull, Sharp estimates and the Dirac operator, *Math. Ann.* **310** (1998) 55–71.
[6] R. Olivier, Über die Denung von Sphärenabbildugen, *Invent. Math.* **1** (1966) 380–390.